WNE Linear Algebra
Sample Exam Questions

Please give reasons for your answers. If the answer is negative, please provide a counterexample.

Question 1.
If $Ax = b$ is a non–homogeneous system of linear equations, is the zero vector a solution of that system?

Question 2.
If vectors $u, v \in \mathbb{R}^4$ are linearly independent and vectors $v, w \in \mathbb{R}^4$ are linearly independent, are vectors $u, v, w \in \mathbb{R}^4$ linearly independent?

Question 3.
If $\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ are linearly dependent, does it follow that vectors $\varphi(v_1), \ldots, \varphi(v_k) \in \mathbb{R}^m$ are linearly dependent?

Question 4.
Is it possible that $A, B \in M(4 \times 4; \mathbb{R}),\ \operatorname{rk} A = \operatorname{rk} B = 2$ but $\operatorname{rk}(A + B) = 3$?

Question 5.
If $A \in M(2 \times 2; \mathbb{R})$, does there exist matrix $B \in M(2 \times 2; \mathbb{R})$ such that

$$B^2 = A?$$

Question 6.
If $A \in M(n \times n; \mathbb{R})$ does it follow that for any $\alpha \in \mathbb{R}$

$$\det(\alpha A) = \alpha^n \det A?$$

Question 7.
If $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ is an endomorphism and

i) $u \in \mathbb{R}^n$ is an eigenvector of φ for the eigenvalue $\lambda \in \mathbb{R}$,
ii) $v \in \mathbb{R}^n$ is an eigenvector of φ for the eigenvalue $\mu \in \mathbb{R}$,

is $u + v \in \mathbb{R}^n$ an eigenvector of φ for the eigenvalue $\lambda + \mu \in \mathbb{R}$?

Question 8.
If matrix $A \in M(n \times n; \mathbb{R})$ is diagonalisable, does it follow that for any matrix $C \in M(n \times n; \mathbb{R})$ such that $\det C \neq 0$, the matrix $C^{-1}AC$ is diagonalisable?

Question 9.
If $u, v \in V$ is a basis of a subspace $V \subset \mathbb{R}^3$ and $w \in V^\perp$ is a basis of the orthogonal completion $V^\perp \subset \mathbb{R}^3$, does it follow that $u, v, w \in \mathbb{R}^3$ is an orthogonal basis of $\mathbb{R}^3$?

Question 10.
If $V \subset \mathbb{R}^n,\ V \neq \mathbb{R}^n$ is a subspace and $P_V\colon \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection onto V, does it follow that 0 is an eigenvalue of P_V?

Question 11.
If $Q\colon \mathbb{R}^2 \to \mathbb{R}$ is a quadratic form, $\mathcal{B} = (v_1, v_2)$ is a basis of $\mathbb{R}^2$ and

$$Q(v_1), Q(v_2) > 0,$$

does it follow that the form Q is positive definite?

Question 12.
If $A, B \in M(n \times n; \mathbb{R})$ are symmetric matrices, is the matrix $A + B^{\mathsf{T}}$ symmetric?

Question 13.
Is it possible for a linear programming problem in 2 variables not to have an optimal solution?